

THE TAYLOR SERIES RELATED TO THE DIFFERENTIAL OF THE EXPONENTIAL MAP

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ABSTRACT. In this paper we study the Taylor series of an operator-valued function related to the differential of the exponential map. For a smooth manifold \mathcal{M} with a torsion-free affine connection the operator $\mathcal{E}_p(v)$ acting on the space $T_p\mathcal{M}$ is defined to be the composition of the differential of the exponential map at $v \in T_p\mathcal{M}$ with parallel transport to p along the geodesic. The Taylor series of \mathcal{E}_p as a function of v is found explicitly in terms of the curvature tensor and its high order covariant derivatives at p .

Key words: affine connection, exponential map, Taylor series.

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1. INTRODUCTION

Let (\mathcal{M}, ∇) be a smooth manifold with a torsion-free affine connection. For a point $p \in \mathcal{M}$, consider the exponential map

$$\text{Exp}_p : T_p\mathcal{M} \rightarrow \mathcal{M}.$$

Let $v \in T_p\mathcal{M}$ be a vector at a fixed point p and γ_v be the corresponding geodesic $\gamma_v(t) = \text{Exp}_p(tv)$. As $T_p\mathcal{M}$ is a linear space, the differential of the exponential map may be considered a linear map of the form

$$d_v \text{Exp}_p : T_p\mathcal{M} \rightarrow T_{\text{Exp}_p v}\mathcal{M}.$$

Denote by

$$\mathcal{I}_p(v) : T_p\mathcal{M} \rightarrow T_{\text{Exp}_p v}\mathcal{M}$$

the operator of parallel transport along the geodesic γ_v . Let \mathcal{E}_p be the map

$$\mathcal{E}_p : T_p\mathcal{M} \rightarrow \text{End}_{\mathbb{R}}(T_p\mathcal{M}),$$

defined by

$$\mathcal{E}_p(v) = \mathcal{I}_p(v)^{-1} \circ d_v \text{Exp}_p.$$

This is a smooth map between two linear spaces, hence its Taylor series is well defined. Our goal is to find this series explicitly in terms of the curvature tensor.

For the best of author's knowledge, the problem has never been considered in this generality. However, for a symmetric space \mathcal{M} the answer is well known [2] (see also [4], Ch. IV, Theorem 4.1). In our terms

$$\mathcal{E}_p(v) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} r_0^k(v); \quad (1)$$

the operator r_0 is defined by

$$r_0(v) : w \mapsto R_p(v, w)v, \quad w \in T_p\mathcal{M},$$

where R is the curvature tensor. Note that this is a very special case, because for a more general manifold the series depends not only on the curvature tensor itself, but on its high order covariant derivatives at the point p as well.

Apparently, the only relevant result known for a general affine (or Riemannian) manifold is the Helgason's formula [3] (in fact, the formula is proved for an *analytic* manifold with an affine connection). The formula is

$$d_v \text{Exp}_p w = \frac{1 - e^{-\text{ad}v^*}}{\text{ad}v^*} w^* \Big|_{\text{Exp}_p v},$$

where the adjoint refers to the Lie algebra of smooth vector fields and v^* denotes the vector field defined by the condition

$$v_{\text{Exp}_p}^* u = \mathcal{I}_p(u)v, \quad u \in T_p\mathcal{M}$$

(the same for w^*). This, of course, is not a Taylor series, and its relation to the problem remains quite obscure.

The author might also mention his own paper [1], where an algorithm of computing the Taylor series of the inverse operator $\mathcal{E}_p(v)^{-1}$ (denoted there by $H(v)$) is proposed. This algorithm, however, is quite involved in comparison with the explicit formula presented here.

Remark. In one respect the Helgason's formula is more general than the results obtained below: it is not necessary to assume the connection to be torsion-free. It should be noted, however, that the differential of the exponential map (as the map itself) does not depend on the torsion part of the affine connection. So, it is natural to restrict ourselves to the torsion-free case. In a more general case the Taylor series contains terms which depend on the torsion tensor, but these terms actually have nothing to do with the exponential map.

2. THE SERIES

Before we can formulate the main result we have to introduce some notation. For $n \geq 0$ and $v \in T_p\mathcal{M}$ denote by $R_{p,v}^{(n)}$ the n -th order covariant derivation of the curvature tensor at the point p in the direction v . That is,

$$R_{p,v}^{(n)} = v^{\otimes n} \cdot (\nabla^n R)_p.$$

(The common notation $\nabla_{v,\dots,v}^n$ for high order covariant derivation in the direction v is not very convenient when n is variable. For this reason we use the contraction notation. The sign \cdot on the right hand side denotes the contraction of the polyvector $v^{\otimes n}$ with the tensor $\nabla^n R$).

For p, v, n as above denote by

$$r_{p,n}(v) \in \text{End}_{\mathbb{R}}(T_p\mathcal{M})$$

the linear operator defined by

$$r_{p,n}(v) : w \mapsto R_{p,v}^{(n)}(v, w)v, \quad w \in T_p\mathcal{M}.$$

For the sake of convenience we usually omit the point and, sometimes, the vector:

$$r_n = r_n(v) = r_{p,n}(v).$$

For example,

$$r_0 : w \mapsto R_p(v, w)v, \quad r_1 : w \mapsto (\nabla_v R)_p(v, w)v,$$

etc. Obviously, r_n as a function of v is homogeneous of degree $n + 2$

$$r_n(tv) = t^{n+2}r_n(v), \quad t \in \mathbb{R}.$$

We also need an appropriate notation for compositions of operators of this kind. Call a finite sequence of nonnegative integers a *list*. The set of all lists (including the empty one) is denoted by Λ . The empty list is denoted by the symbol \emptyset ; to write down a nonempty list we use square brackets; for example,

$$\nu = [2, 0, 1] \in \Lambda.$$

For every list $\nu \in \Lambda$ there is a corresponding operator $r_\nu \in \text{End}_{\mathbb{R}}(T_p\mathcal{M})$ which is a composition of simple operators r_n . Namely, if $\nu = [n_1, n_2, \dots, n_k]$ then

$$r_\nu = r_{n_1}r_{n_2} \dots r_{n_k}.$$

By definition, $r_\emptyset = \mathbb{1}$.

We shall need three number functions on lists: the factorial $\nu!$, the degree $|\nu|$ and the denominator c_ν . By definition, $\emptyset! = 1, |\emptyset| = 0$. For $\nu = [n_1, n_2, \dots, n_k]$ we define the factorial and the degree as follows

$$\nu! = \prod_{j=1}^k (n_j!), \quad |\nu| = 2k + \sum_{j=1}^k n_j.$$

Obviously,

$$r_\nu(tv) = t^{|\nu|}r_\nu(v), \quad t \in \mathbb{R},$$

which is where the term “degree” comes from.

The denominator is defined by $c_\emptyset = 1$ and a recurrent relation

$$c_\nu = |\nu|(|\nu| + 1)c_{\nu'}, \quad \nu \in \Lambda \setminus \{\emptyset\},$$

where ν' is obtained from ν by omitting the first element from the list. For example, $|[2, 0, 1]| = 2 \cdot 3 + 2 + 0 + 1 = 9$, hence

$$c_{[201]} = 9 \cdot 10c_{[01]} = 90 \cdot 30c_{[1]} = 90 \cdot 30 \cdot 12c_\emptyset = 32400.$$

Now we are able to formulate the main result.

Theorem *Let (\mathcal{M}, ∇) be a smooth manifold with a torsion-free affine connection. Let $p \in \mathcal{M}$ and $v \in T_p \mathcal{M}$. Then for any $n \geq 0$ the following equality holds*

$$\left. \frac{1}{n!} \frac{d^n}{dt^n} \mathcal{E}_p(tv) \right|_{t=0} = \sum_{|\nu|=n} \frac{1}{\nu! c_\nu} r_\nu(v), \quad (2)$$

where the sum on the right hand side is taken over all the lists $\nu \in \Lambda$ of degree n .

In other words, we have the Taylor series

$$\mathcal{E}_p(v) = \sum_{\nu \in \Lambda} \frac{1}{\nu! c_\nu} r_\nu(v). \quad (3).$$

For example, there are 13 lists of degree not greater than 6, namely $\emptyset, [0], [1], [2], [0, 0], [3], [1, 0], [0, 1], [4], [2, 0], [1, 1], [0, 2], [0, 0, 0]$. Computing the corresponding coefficients, we have the

Corollary *The function \mathcal{E}_p can be expressed as follows*

$$\begin{aligned} \mathcal{E}_p(v) = & \mathbb{1} + \frac{1}{6}r_0 + \frac{1}{12}r_1 + \frac{1}{40}r_2 + \frac{1}{120}r_0^2 + \frac{1}{180}r_3 + \frac{1}{180}r_1r_0 + \frac{1}{360}r_0r_1 + \frac{1}{1008}r_4 + \\ & + \frac{1}{504}r_2r_0 + \frac{1}{504}r_1^2 + \frac{1}{1680}r_0r_2 + \frac{1}{5040}r_0^3 + \rho_7(v), \end{aligned}$$

where $\rho_7(v) = O(|v|^7)$ as $v \rightarrow 0$.

Note that for a list $\nu = [0, 0, \dots, 0]$ which consists of k zeros we have $|\nu| = 2k$, hence

$$c_\nu = 2k(2k+1)c_{\nu'} = (2k+1)!.$$

Thus, the coefficient at the term $r_\nu = r_0^k$ in the Taylor series is equal to $1/(2k+1)!$, in agreement with (1).

It may be shown that there are no nontrivial algebraic relations between the operators r_n on a general manifold. So, the series (3) is unique as a formal series in non-commutative variables r_n .

3. PARALLEL TRANSPORT OF THE CURVATURE TENSOR

We shall need some properties of the covariant derivation along a geodesic. For our purpose it is convenient to consider a connection on a vector bundle instead of an affine connection (which is a connection on the tangent bundle). Let $E \rightarrow \mathcal{M}$ be a smooth vector bundle on a smooth manifold. The space of smooth sections of E is denoted by $C^\infty(\mathcal{M}, E)$. The bundle is provided with a connection ∇ , which is a linear map

$$\nabla : C^\infty(\mathcal{M}, E) \rightarrow C^\infty(\mathcal{M}, T^*\mathcal{M} \otimes E),$$

satisfying the Leibniz rule

$$\nabla f u = df \otimes u + f \nabla u, \quad f \in C^\infty(\mathcal{M}), u \in C^\infty(\mathcal{M}, E).$$

Note that the manifold itself is not supposed to have a connection for now.

Let $\gamma : I \rightarrow \mathcal{M}$ be a smooth curve, where $I \subset \mathbb{R}$ is an interval. The induced connection on the restricted bundle $\gamma^*E \rightarrow I$ is usually denoted

by the sign D . It is convenient to use this connection in the form of a differential operator $\frac{D}{dt}$, where $t \in I$ is the parameter on the curve. This operator is called a *covariant derivation along the curve* γ . In other words, for any section $u \in C^\infty(\mathcal{M}, E)$ we have the equality

$$\frac{D}{dt}\gamma^*u = \dot{\gamma} \cdot \gamma^*\nabla u, \quad (4)$$

where

$$\dot{\gamma}(t) = \frac{d}{dt}\gamma(t) \in T_{\gamma(t)}\mathcal{M}, \quad t \in I.$$

To be less pedantic, one may write (4) as an operator identity

$$\frac{D}{dt} = \nabla_{\dot{\gamma}}.$$

We shall need the following simple, but useful

Lemma 1 *Let (\mathcal{M}, ∇) be a smooth manifold with an affine connection and $E \rightarrow \mathcal{M}$ be a smooth vector bundle with a connection. Let $\gamma : I \rightarrow \mathcal{M}$ be a geodesic. Then for any $n \geq 1$ and any smooth section u of the bundle E*

$$\frac{D^n}{dt^n}\gamma^*u = \dot{\gamma}^{\otimes n} \cdot \gamma^*\nabla^n u, \quad (5)$$

where $\frac{D}{dt}$ is the covariant derivation along γ .

The equality (5) may also be written in the operator form

$$\frac{D^n}{dt^n} = \dot{\gamma}^{\otimes n} \cdot \nabla^n.$$

Apparently, this formula is well known, but the author does not know a proper reference. For this reason, we present here a proof. Note that for $n > 1$ the section $\nabla^n u$ on the right hand side of (5) depends on the affine connection on the manifold while the left hand side depends on the curve γ only.

Proof. It is an almost trivial fact that the derivation $\frac{D}{dt}$ can be canonically extended to the products of E with tensor bundles on \mathcal{M} and inherits all the common properties of the covariant derivation. In particular, it is compatible with contraction. For $n = 1$ the formula (5) is the definition of $\frac{D}{dt}$. If it is valid for some n , then we have

$$\frac{D^{n+1}}{dt^{n+1}}\gamma^*u = \frac{D}{dt}\frac{D^n}{dt^n}\gamma^*u = \frac{D}{dt}\dot{\gamma}^{\otimes n} \cdot \gamma^*\nabla^n u = \left(\frac{D}{dt}\dot{\gamma}^{\otimes n}\right) \cdot \gamma^*\nabla^n u + \dot{\gamma}^{\otimes n} \cdot \frac{D}{dt}\gamma^*\nabla^n u.$$

By assumption, γ is a geodesic, hence

$$\frac{D}{dt}\dot{\gamma} = 0$$

and the first term on the right hand side vanishes. Applying (4) to the section $\nabla^n u \in C^\infty(\mathcal{M}, T^*\mathcal{M}^{\otimes n} \otimes E)$, we have

$$\frac{D}{dt}\gamma^*\nabla^n u = \dot{\gamma} \cdot \gamma^*\nabla^{n+1}u.$$

Thus,

$$\frac{D^{n+1}}{dt^{n+1}} \gamma^* u = \dot{\gamma}^{\otimes n+1} \cdot \gamma^* \nabla^{n+1} u,$$

and the equality (5) follows by induction. \square

The covariant derivation is closely related to parallel transport. As above, denote by $\mathcal{I}_p = \mathcal{I}_{p,E}$ the operator of parallel transport of the form

$$\mathcal{I}_p(v) : E_p \rightarrow E_{\text{Exp}_p v}$$

along the geodesic $\gamma_v(t) = \text{Exp}_p(tv)$. By the definition of parallel transport,

$$\frac{D}{dt} \mathcal{I}_p(tv) z = 0,$$

where $\frac{D}{dt}$ is the covariant derivation along γ_v and $z \in E_p$ is a constant. In a more general case, when $z = z(t)$ depends on the parameter t , we have the operator equality

$$\frac{D}{dt} \circ \mathcal{I}_p(tv) = \mathcal{I}_p(tv) \circ \frac{d}{dt}, \quad (6)$$

which will be of use below.

After the above preliminaries we have come to the matter. Consider the operator $\mathcal{R}_p(v) \in \text{End}_{\mathbb{R}}(T_p \mathcal{M})$, defined by

$$\mathcal{R}_p(v)w = \mathcal{I}_p(v)^{-1} R_{\text{Exp}_p v}(\mathcal{I}_p(v)v, \mathcal{I}_p(v)w) \mathcal{I}_p(v)v, \quad w \in T_p \mathcal{M}, \quad (7)$$

where R_x denotes the curvature tensor at a point $x \in \mathcal{M}$. As is well known, parallel transport is compatible with tensor operations. Thus, this operator can also be written in the form

$$\mathcal{R}_p(v)w = (\mathcal{I}_p(v)^{-1} R_{\text{Exp}_p v})(v, w)v.$$

In the latter equality the parallel transport operator $\mathcal{I}_p(v)$ is applied to the curvature tensor, i.e. it acts on the bundle $T_3^1 \mathcal{M}$.

Lemma 2 For $v \in T_p \mathcal{M}$ and $n \geq 0$,

$$\left. \frac{d^n}{dt^n} \mathcal{R}_p(tv) \right|_{t=0} = n(n-1) r_{n-2}(v). \quad (8)$$

In fact, r_{-1} and r_{-2} are not defined, but it is convenient to consider them arbitrary operators. So, for $n = 0$ and $n = 1$ the right hand side of (8) is zero.

Proof. Consider the geodesic $\gamma = \gamma_v$. For $w \in T_p \mathcal{M}$ we have

$$\mathcal{R}_p(tv)w = t^2 (\mathcal{I}_p(tv)^{-1} R_{\gamma(t)})(v, w)v,$$

hence

$$\left. \frac{d^n}{dt^n} \mathcal{R}_p(tv) \right|_{t=0} w = n(n-1) \left(\frac{d^{n-2}}{dt^{n-2}} \mathcal{I}_p(tv)^{-1} R_{\gamma(t)} \right) \Big|_{t=0} (v, w)v.$$

By (6),

$$\frac{d^{n-2}}{dt^{n-2}} \mathcal{I}_p(tv)^{-1} R_{\gamma(t)} = \mathcal{I}_p(tv)^{-1} \frac{D^{n-2}}{dt^{n-2}} R_{\gamma(t)}.$$

By Lemma 1,

$$\left. \frac{D^{n-2}}{dt^{n-2}} R_{\gamma(t)} \right|_{t=0} = v^{\otimes n-2} \cdot (\nabla^{n-2} R)_p = R_{p,v}^{(n-2)}.$$

Taking into account the equality $\mathcal{I}_p(0) = \mathbb{1}$ and the definition of r_{n-2} , we have (8). \square

4. THE JACOBI FIELD

For given vectors $v, w \in T_p \mathcal{M}$ consider a family of geodesics

$$\gamma_s(t) = \text{Exp}_p(tv + tsw),$$

parametrized by a real number s taken from a neighbourhood of zero. Denote by $\gamma = \gamma_0 = \gamma_v$ the geodesic corresponding to $s = 0$. Let $J \in C^\infty(I, \gamma^* T\mathcal{M})$ be the vector field defined by

$$J_t = \left. \frac{d}{ds} \gamma_s(t) \right|_{s=0} \in T_{\gamma(t)} \mathcal{M},$$

which is equivalent to

$$J_t = d_{tv} \text{Exp}_p tw.$$

It is well known that a field of this kind is a Jacobi field [5]. This means that it satisfies the differential equation

$$\frac{D^2}{dt^2} J_t = R_{\gamma(t)}(\dot{\gamma}(t), J_t) \dot{\gamma}(t). \quad (9)$$

Lemma 3 *For any $v \in T_p \mathcal{M}$ the operator $\mathcal{E}_p(tv)$ satisfies the differential equation*

$$\left(t^2 \frac{d^2}{dt^2} + 2t \frac{d}{dt} \right) \mathcal{E}_p(tv) = \mathcal{R}_p(tv) \mathcal{E}_p(tv). \quad (10)$$

Proof. By the definition of \mathcal{E}_p ,

$$\mathcal{E}_p(tv)tw = \mathcal{I}_p(tv)^{-1} \circ d_{tv} \text{Exp}_p tw = \mathcal{I}_p(tv)^{-1} J_t.$$

Thus, by (6) and (9),

$$\left(t \frac{d^2}{dt^2} + 2 \frac{d}{dt} \right) \mathcal{E}_p(tv)w = \frac{d^2}{dt^2} \mathcal{E}_p(tv)tw = \mathcal{I}_p(tv)^{-1} \frac{D^2}{dt^2} J_t = \mathcal{I}_p(tv)^{-1} R_{\gamma(t)}(\dot{\gamma}(t), J_t) \dot{\gamma}(t).$$

On the other hand, by the definition of the operator \mathcal{R} , we have the equality

$$\mathcal{R}_p(tv) \mathcal{E}_p(tv)w = t \mathcal{I}_p(tv)^{-1} R_{\gamma(t)}(\dot{\gamma}(t), J_t) \dot{\gamma}(t).$$

Comparing these equalities and taking into account that they are valid for any choice of w , we have (10). \square

Proof of Theorem. For $n \geq 0$ denote the left hand side of (2) by

$$\mathcal{E}_n = \left. \frac{1}{n!} \frac{d^n}{dt^n} \mathcal{E}_p(tv) \right|_{t=0}.$$

The equality

$$\mathcal{E}_n = \sum_{|\nu|=n} \frac{1}{\nu! c_\nu} r_\nu \quad (11)$$

for $n = 0$ and $n = 1$ can be verified directly:

$$\mathcal{E}_0 = r_\emptyset = \mathbb{1}, \mathcal{E}_1 = 0.$$

Taking the n -th order derivative of the both sides of (10) at $t = 0$, we have by Lemma 2 the equality

$$n(n+1)n!\mathcal{E}_n = \sum_{k=0}^n \binom{n}{k} k(k-1)r_{k-2} \cdot (n-k)!\mathcal{E}_{n-k}.$$

For $n \geq 2$ the equality takes the form

$$\mathcal{E}_n = \frac{1}{n(n+1)} \sum_{m=0}^{n-2} \frac{1}{m!} r_m \mathcal{E}_{n-m-2}. \quad (12)$$

By induction, we may assume that the formula (11) is valid for the operators \mathcal{E}_{n-m-2} on the right hand side of (12). We have then the equality

$$\mathcal{E}_n = \sum_{m=0}^{n-2} \sum_{|\mu|=n-m-2} \frac{1}{m!\mu!n(n+1)c_\mu} r_m r_\mu.$$

Denote $\nu = [m, \mu]$ (then $\nu' = \mu$). One can see that the double sum in the latter equality can be replaced by a single sum taken over the lists ν of degree n . Taking into account the obvious equalities

$$\nu! = m!\mu!, c_\nu = n(n+1)c_\mu, r_\nu = r_m r_\mu,$$

we obtain (11). □

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